

Blow-up collocation solutions of some Volterra integral equations

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Abstract

In this paper we use collocation methods for detecting blow-up solutions of nonlinear homogeneous Volterra-Hammerstein integral equations. To do this, first we generalize the results on existence and uniqueness given in a previous paper in order to consider also non locally bounded kernels, which are present in many cases of blow-up problems. Next, we introduce the concept of “blow-up collocation solution” and analyze numerically some blow-up time estimates using collocation methods in particular examples where the previous results about existence and uniqueness can be applied. Finally, we discuss the relationships between necessary conditions for blow-up of collocation solutions and exact solutions.

1 Introduction

Some engineering and industrial problems are described by explosive phenomena which are modeled by nonlinear integral equations whose solutions exhibit blow-up at finite time (see [1, 2] and references therein). Many authors have studied necessary and sufficient conditions for the existence of such blow-up time. In particular, in [3, 4, 5, 6, 7] equation

$$y(t) = \int_0^t K(t, s) G(y(s)) ds, \quad t \in I := [0, T]. \quad (1)$$

was considered. In these works, conditions for the existence of a finite blow-up time, as well as upper and lower estimates of it, were given, although, in some cases, they were not very accurate.

A way of improving these estimations is to study numerical approximations of the solution; in this aspect, collocation methods have proven to be a very suitable technique for approximating nonlinear integral equations, because of its stability and accuracy (see [8]). Hence, the aim of this paper is to check the usefulness of collocation methods for detecting blow-up solutions of the nonlinear HVHIE (homogeneous Volterra-Hammerstein integral equation) given by equation (1).

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In a previous work [9], we used collocation methods for approximating the nontrivial solutions of (1). In particular we studied the collocation solutions of the implicitly linear HVIE (homogeneous Volterra integral equation)

$$z(t) = G((\mathcal{V}z)(t)) = G\left(\int_0^t K(t, s) z(s) ds\right), \quad t \in I, \quad (2)$$

being \mathcal{V} the linear *Volterra operator*, and assuming that some *general conditions* were held. In particular, it was required that the kernel K was locally bounded; nevertheless, non locally bounded kernels appear in many cases of blow-up solutions (see [7]) and so, in the present work, we shall first replace this condition with another that does not exclude this kind of kernels. Hence, the new *general conditions* that we are going to impose (even if it is not explicitly mentioned) are:

- **Over K .** The *kernel* $K : \mathbb{R}^2 \rightarrow [0, +\infty[$ has its support in $\{(t, s) \in \mathbb{R}^2 : 0 \leq s \leq t\}$.

For every $t > 0$, the map $s \mapsto K(t, s)$ is locally integrable, and $\mathcal{K}(t) := \int_0^t K(t, s) ds$ is a strictly increasing function.

New condition: $\lim_{t \rightarrow a^+} \int_a^t K(t, s) ds = 0$ for all $a \geq 0$.

Note that for convolution kernels (i.e. $K(t, s) = k(t - s)$) this new condition always holds, because obviously k is locally integrable and then

$$\lim_{t \rightarrow a^+} \int_a^t k(t - s) ds = \lim_{t \rightarrow a^+} \int_0^{t-a} k(s) ds = \lim_{t \rightarrow 0^+} \int_0^t k(s) ds = 0.$$

- **Over G .** The *nonlinearity* $G : [0, +\infty[\rightarrow [0, +\infty[$ is a continuous, strictly increasing function, and $G(0) = 0$.

There is a one-to-one correspondence between solutions of (1) and (2) (see [8, 10]). Particularly, if z is a solution of (2), then $y := \mathcal{V}z$ is a solution of (1). Moreover, $y = G^{-1} \circ z$, since G is injective.

The paper is structured as follows: in Section 2 we introduce briefly the basic definitions and state the notation. Next, we devote Section 3 to generalize the results about existence and uniqueness of nontrivial collocation solutions obtained in [9] to the new *general conditions*. In Section 4, we introduce the concept of “blow-up collocation solution” and analyse numerically some blow-up time estimates using collocation methods in particular examples where the previous results about existence and uniqueness can be applied. Finally, in Section 5, we discuss the relationships between necessary conditions for blow-up of collocation solutions and exact solutions.

2 Collocation problems for implicitly linear HVIEs

Following the notation of [8], a *collocation problem* for equation (2) is given by a mesh $I_h := \{t_n : 0 = t_0 < t_1 < \dots < t_N = T\}$ and a set of m *collocation parameters* $0 \leq c_1 < \dots < c_m \leq 1$. We denote $h_n := t_{n+1} - t_n$ ($n = 0, \dots, N-1$) and the quantity $h := \max\{h_n : 0 \leq n \leq N-1\}$ is called the *stepsize*. Moreover, the *collocation points* are given by $t_{n,i} := t_n + c_i h_n$ ($i = 1, \dots, m$), and the set of collocation points is denoted by X_h (see [8, 11]). A *collocation solution* z_h is then given by the *collocation equation*

$$z_h(t) = G\left(\int_0^t K(t, s) z_h(s) ds\right), \quad t \in X_h,$$

where z_h is in the space of piecewise polynomials of degree less than m .

Remark 2.1 From now on, a “collocation problem” or a “collocation solution” will be always referred to the implicitly linear HVIE (2). So, if we want to obtain an estimation of a solution of the nonlinear HVHIE (1), then we have to consider $y_h := \mathcal{V}z_h$.

Remark 2.2 At the beginning of Section 4, we define the concepts of “blow-up collocation problems” and “blow-up collocation solutions”. To do this, we extend the concept of “collocation solution” to meshes I_h with infinite points. Taking this into account, if z_h is a blow-up collocation solution, then y_h also blows up at the same blow-up time.

As it is stated in [8], a collocation solution z_h is completely determined by the coefficients $Z_{n,i} := z_h(t_{n,i})$ ($n = 0, \dots, N-1$) ($i = 1, \dots, m$), since $z_h(t_n + vh_n) = \sum_{j=1}^m L_j(v) Z_{n,j}$ for all $v \in]0, 1]$, where $L_1(v) := 1$, $L_j(v) := \prod_{k \neq j}^m \frac{v - c_k}{c_j - c_k}$ ($j = 2, \dots, m$) are the Lagrange fundamental polynomials with respect to the collocation parameters. The values of $Z_{n,i}$ are given by the system

$$Z_{n,i} = G \left(F_n(t_{n,i}) + h_n \sum_{j=1}^m B_n(i, j) Z_{n,j} \right), \quad (3)$$

where

$$B_n(i, j) := \int_0^{c_i} K(t_{n,i}, t_n + sh_n) L_j(s) ds$$

and

$$F_n(t) := \int_0^{t_n} K(t, s) z_h(s) ds. \quad (4)$$

The term $F_n(t_{n,i})$ is called the *lag term*.

3 Existence and uniqueness of nontrivial collocation solutions

Given a kernel K , a nonlinearity G and some collocation parameters $\{c_1, \dots, c_m\}$, in [9] there were defined three kinds of existence of nontrivial collocation solutions (of the corresponding collocation problem) in an interval $I = [0, T]$ using a mesh I_h : **existence near zero** (collocation solutions can always be extended a bit more, but it is not ensured the existence of nontrivial collocation solutions for arbitrarily large T), **existence for fine meshes** (there exists a fine enough mesh for which there is a nontrivial collocation solution), and **unconditional existence** (given any mesh, there always exists a nontrivial collocation solution). For these two last kinds of existence, it is ensured the existence of nontrivial collocation solutions in any interval, and hence, there is no blow-up.

In [9], two particular cases of collocation problems were considered:

- Case 1: $m = 1$ with $c_1 > 0$.
- Case 2: $m = 2$ with $c_1 = 0$.

In these cases, the system (3) is reduced to a single nonlinear equation, whose solution is given by the fixed points of $G(\alpha + \beta y)$ for some α, β .

The proofs of the results obtained in [9] rely on the locally boundedness of the kernel K . Therefore, we will revise them, imposing the new *general conditions* and avoiding other hypotheses that turn out to be unnecessary. For example, we extend the results to decreasing convolution kernels, among others.

As in the previous work [9], we will need a lemma on nonlinearities G satisfying the *general conditions*:

Lemma 3.1 *The following statements are equivalent to the statement that $\frac{G(y)}{y}$ is unbounded (in $]0, +\infty[$):*

- (i) *There exists $\beta_0 > 0$ such that $G(\beta y)$ has nonzero fixed points for all $0 < \beta \leq \beta_0$.*
- (ii) *Given $A \geq 0$, there exists $\beta_A > 0$ such that $G(\alpha + \beta y)$ has nonzero fixed points for all $0 \leq \alpha \leq A$ and for all $0 < \beta \leq \beta_A$.*

3.1 Case 1: $m = 1$ with $c_1 > 0$

First, we shall consider $m = 1$ with $c_1 > 0$. Equations (3) are then reduced to

$$Z_{n,1} = G(F_n(t_{n,1}) + h_n B_n Z_{n,1}) \quad (n = 0, \dots, N-1), \quad (5)$$

where

$$B_n := B_n(1, 1) = \int_0^{c_1} K(t_{n,1}, t_n + sh_n) \, ds \quad (6)$$

and the lag terms $F_n(t_{n,1})$ are given by (4) with $i = 1$. Note that $B_n > 0$ because the integrand in (6) is strictly positive almost everywhere in $]0, c_1[$.

Remark 3.1 In [9], it is ensured that $\lim_{h_n \rightarrow 0^+} h_n B_n = 0$ taking into account (6) and the old *general conditions* imposed over K . Thus, we must ensure that the limit also vanishes considering the new *general conditions*. In fact, with a change of variable, we can express (6) as

$$B_n = \frac{1}{h_n} \int_{t_n}^{t_{n,1}} K(t_{n,1}, s) \, ds. \quad (7)$$

Hence, by (7) and the new condition, we have

$$\lim_{h_n \rightarrow 0^+} h_n B_n = \lim_{t \rightarrow t_n^+} \int_{t_n}^t K(t, s) \, ds = 0,$$

since $t_{n,1} = t_n + c_1 h_n$.

Now, we are in position to give a characterization of the existence near zero of nontrivial collocation solutions:

Proposition 3.1 *There is existence near zero if and only if $\frac{G(y)}{y}$ is unbounded.*

Proof.

(\Leftarrow) Let us prove that if $\frac{G(y)}{y}$ is unbounded, then there is existence near zero. So, we are going to prove by induction over n that there exist $H_n > 0$ ($n = 0, \dots, N-1$) such that if $0 < h_n \leq H_n$ then there exist solutions of the system (5) with $Z_{0,1} > 0$:

- For $n = 0$, taking into account Remark 3.1 and Lemma 3.1-(i) (see below), we choose a small enough $H_0 > 0$ such that $0 < h_0 B_0 \leq \beta_0$ for all $0 < h_0 \leq H_0$. So, since the lag term is 0, we can apply Lemma 3.1-(i) to the equation (5), concluding that there exist strictly positive solutions for $Z_{0,1}$.
- Let us suppose that, choosing one of those $Z_{0,1}$ and given $n > 0$, there exist $H_1, \dots, H_{n-1} > 0$ such that if $0 < h_i \leq H_i$ ($i = 1, \dots, n-1$) then there exist coefficients $Z_{1,1}, \dots, Z_{n-1,1}$ fulfilling the equation (5). Note that these coefficients are strictly positive, and hence, it is guaranteed that the corresponding collocation solution z_h is (strictly) positive in $]0, t_n]$.

- Finally, we are going to prove that there exists $H_n > 0$ such that if $0 < h_n \leq H_n$ then there exists $Z_{n,1} > 0$ fulfilling the equation (5) with the previous coefficients $Z_{0,1}, \dots, Z_{n-1,1}$:

Let us define

$$A := \max \{F_n(t_n + c_1\epsilon) : 0 \leq \epsilon \leq 1\}.$$

Note that A exists because $s \mapsto K(t, s)$ is locally integrable for all $t > 0$, and z_h is bounded in $[0, t_n]$; hence, F_n (see (4)) is bounded in $[t_n, t_n + c_1]$. Moreover, $A \geq 0$ because K and z_h are positive functions.

So, applying Lemma 3.1-(ii) (see below), there exists $\beta_A > 0$ such that $G(\alpha + \beta y)$ has nonzero (strictly positive) fixed points for all $0 \leq \alpha \leq A$ and for all $0 < \beta \leq \beta_A$. On one hand, taking into account Remark 3.1, we choose a small enough $0 < H_n \leq 1$ such that $0 < h_n B_n \leq \beta_A$ for all $0 < h_n \leq H_n$; on the other hand, choosing one of those h_n , we have $0 \leq F(t_{n,1}) \leq A$. Hence, we obtain the existence of $Z_{n,1}$ as the strictly positive fixed point of $G(F(t_{n,1}) + h_n B_n y)$.

(\Rightarrow) For proving the other condition, we use Lemma 3.1-(i), taking into account Remark 3.1. ■

Remark 3.2 A similar result was proved in [9] with the additional hypothesis “ $K(t, s) \leq K(t', s)$ for all $0 \leq s \leq t < t'$ ” (or “ k is increasing” for the particular case of convolution kernels $K(t, s) = k(t - s)$). We have shown that this hypothesis is not needed and the difference between the proofs is the choice of A .

Taking into account Remark 3.1, we can adapt the results given in [9] about sufficient conditions on existence for fine meshes and unconditional existence to our new *general conditions*. This will help us to identify collocation problems without blow-up (see Proposition 3.2 below). But first we need to recall the following definition.

Definition 3.1 We say that a property \mathcal{P} holds *near zero* if there exists $\epsilon > 0$ such that \mathcal{P} holds on $]0, \delta[$ for all $0 < \delta < \epsilon$. On the other hand, we say that \mathcal{P} holds *away from zero* if there exists $\tau > 0$ such that \mathcal{P} holds on $]t, +\infty[$ for all $t > \tau$.

The concept “away from zero” was originally defined in a more restrictive way in [9]; nevertheless both definitions are appropriate for our purposes.

Proposition 3.2 Let $\frac{G(y)}{y}$ be unbounded near zero.

- If K is a convolution kernel and $\frac{G(y)}{y}$ is bounded away from zero, then there is existence for fine meshes.
- If $\frac{y}{G(y)}$ is unbounded away from zero, then there is unconditional existence.

If, in addition, $\frac{G(y)}{y}$ is a strictly decreasing function, then there is at most one nontrivial collocation solution.

Remark 3.3 The proof is analogous to the one given in [9]. Note that the property “ $\frac{y}{G(y)}$ is unbounded away from zero” appears there as “there exists a sequence $\{y_n\}_{n=1}^{+\infty}$ of positive real numbers and divergent to $+\infty$ such that $\lim_{n \rightarrow +\infty} \frac{G(y_n)}{y_n} = 0$ ”, but both are equivalent.

3.2 Case 2: $m = 2$ with $c_1 = 0$

Considering $m = 2$ with $c_1 = 0$, we have to solve the following equations:

$$Z_{n,1} = G(F_n(t_{n,1})) \quad (8)$$

$$Z_{n,2} = G(F_n(t_{n,2}) + h_n B_n(2, 1) Z_{n,1} + h_n B_n(2, 2) Z_{n,2}), \quad (9)$$

for $n = 0, \dots, N - 1$, where

$$B_n(2, j) = \int_0^{c_2} K(t_{n,2}, t_n + sh_n) L_j(s) ds, \quad (j = 1, 2) \quad (10)$$

and $F_n(t_{n,i})$ ($i = 1, 2$) are given by (4). Note that $B_n(2, j) > 0$ for $j = 1, 2$, because the integrand in (10) is strictly positive almost everywhere in $]0, c_2[$.

Remark 3.4 As in the first case, we have to ensure that $\lim_{h_n \rightarrow 0^+} h_n B_n(2, j) = 0$ for $j = 1, 2$, taking into account the new *general conditions* (see Remark 3.1). Actually, since $0 < L_j(s) < 1$ for $s \in]0, c_2[$, we have

$$0 \leq h_n B_n(2, j) \leq h_n \int_0^{c_2} K(t_{n,2}, t_n + sh_n) ds = \int_{t_n}^{t_{n,2}} K(t_{n,2}, s) ds. \quad (11)$$

Hence, by (11) and the new condition, the following inequalities are fulfilled:

$$0 \leq \lim_{h_n \rightarrow 0^+} h_n B_n(2, j) \leq \lim_{t \rightarrow t_n^+} \int_{t_n}^t K(t, s) ds = 0,$$

Analogously to the previous case, we present a characterization of the existence near zero of nontrivial collocation solutions:

Proposition 3.3 *Let the map $t \mapsto K(t, s)$ be continuous in $]s, \epsilon[$ for some $\epsilon > 0$ and for all $0 \leq s < \epsilon$ (this hypothesis can be removed if $c_2 = 1$). Then there is existence near zero if and only if $\frac{G(y)}{y}$ is unbounded.*

Proof.

(\Leftarrow) Let us prove that if $\frac{G(y)}{y}$ is unbounded, then there is existence near zero. So, we are going to prove by induction over n that there exist $H_n > 0$ ($n = 0, \dots, N - 1$) such that if $0 < h_n \leq H_n$ then there exist solutions of the system (9) with $Z_{0,2} > 0$:

- For $n = 0$, taking into account Remark 3.4 and Lemma 3.1-(i), we choose a small enough $H_0 > 0$ such that $0 < h_0 B_0(2, 2) \leq \beta_0$ for all $0 < h_0 \leq H_0$. So, since the lag terms are 0 and $Z_{0,1} = G(0) = 0$, we can apply Lemma 3.1-(i) to the equation (9), concluding that there exist strictly positive solutions for $Z_{0,2}$.
- Given $n > 0$, let us suppose that, choosing one of those $Z_{0,2}$, there exist $H_1, \dots, H_{n-1} > 0$ such that if $0 < h_i \leq H_i$ ($i = 1, \dots, n - 1$) then there exist coefficients $Z_{1,2}, \dots, Z_{n-1,2}$ fulfilling the equation (9). Moreover, let us suppose that z_h is positive in $[0, t_n]$, i.e. these coefficients satisfy $Z_{l,2} \geq (1 - c_2) Z_{l,1} \geq 0$ for $l = 1, \dots, n - 1$, where $Z_{l,1}$ is given by (8).
- Finally, we are going to prove that there exists $H_n > 0$ such that if $0 < h_n \leq H_n$ then there exists $Z_{n,2} > 0$ fulfilling the equation (9) with the previous coefficients, and z_h is positive in $[0, t_{n+1}]$, i.e. $Z_{n,2} \geq (1 - c_2) Z_{n,1} \geq 0$:

Let us define

$$A := \max \left\{ F_n(t_n + c_2 \epsilon) + \epsilon G(F_n(t_n + c_1 \epsilon)) \int_0^{c_2} K(t_{n,2}, t_n + sh_n) L_1(s) ds : 0 \leq \epsilon \leq 1 \right\}. \quad (12)$$

Note that A exists and $A \geq 0$ because $s \mapsto K(t, s)$ is locally integrable for all $t > 0$, z_h is bounded and positive in $[0, t_n]$, the nonlinearity G is bounded and positive, and the polynomial L_1 is bounded and positive in $[0, c_2]$.

So, applying Lemma 3.1-(ii), there exists $\beta_A > 0$ such that $G(\alpha + \beta y)$ has nonzero (strictly positive) fixed points for all $0 \leq \alpha \leq A$ and for all $0 < \beta \leq \beta_A$. On one hand, taking into account Remark 3.4, we choose a small enough $0 < H_n \leq 1$ such that $0 < h_n B_n(2, 2) \leq \beta_A$ for all $0 < h_n \leq H_n$; on the other hand, taking into account (4), the lag terms $F_n(t_{n,i})$ are positive for $i = 1, 2$, because K and z_h are positive. Therefore, by (8), $Z_{n,1} = G(F_n(t_{n,1}))$ is positive, because G is positive. Moreover, $h_n B_n(2, 1)$ is positive. So,

$$0 \leq F_n(t_{n,2}) + h_n B_n(2, 1) Z_{n,1} \leq A.$$

Hence, we obtain the existence of $Z_{n,2}$ as the strictly positive fixed point of the function $G(\alpha + \beta y)$ where $\alpha := F_n(t_{n,2}) + h_n B_n(2, 1) Z_{n,1}$ and $\beta := h_n B_n(2, 2)$.

Concluding, we have to check that $Z_{n,2} \geq (1 - c_2) Z_{n,1}$. On one hand,

$$Z_{n,2} = G\left(F_n(t_{n,2}) + h_n \sum_{j=1}^2 B_n(2, j) Z_{n,j}\right) \geq G(F_n(t_{n,2}))$$

because $B_n(2, j) Z_{n,j} \geq 0$ for $j = 1, 2$. On the other hand, since $t \mapsto K(t, s)$ is continuous in $]s, \epsilon[$ for some $\epsilon > 0$ and for all $0 \leq s < \epsilon$, we can suppose that the stepsize is small enough for $t_{n,1} < \epsilon$, and then $F_n(t)$ is continuous in $t_{n,1}$ (see 4). Hence, since G is continuous, $G(F_n(t))$ is also continuous in $t_{n,1}$, i.e. $\lim_{h_n \rightarrow 0^+} G(F_n(t_{n,2})) = G(F_n(t_{n,1}))$. Therefore, choosing a small enough h_n , we have

$$G(F_n(t_{n,2})) \geq (1 - c_2) G(F_n(t_{n,1})) = (1 - c_2) Z_{n,1}.$$

(\Rightarrow) For proving the other condition, we use Lemma 3.1-(i), taking into account Remark 3.4. ■

Remark 3.5 A similar result was proved in [9], but the hypothesis on the kernel was “ $K(t, s) \leq K(t', s)$ for all $0 \leq s \leq t < t'$ ”, and so, unlike Proposition 3.3, it could not be applied to decreasing convolution kernels (for example). Again, the main difference between both proofs lies in the choice of A .

Note that for convolution kernels $K(t, s) = k(t - s)$, the hypothesis on K is equivalent to say that k is continuous near zero, which is a fairly weak hypothesis. On the other hand, for general kernels, this hypothesis is only needed for ensuring that the mapping $t \mapsto K(t, s)$ is continuous in $t_{n,1}$. Therefore, if it is not continuous, it is important to choose a step h_i such that $t \mapsto K(t, s)$ is continuous at the collocation points $t_{i,1}$ ($i = 0, \dots, N - 1$). More specifically, the hypothesis on K is only needed for checking that $Z_{n,2} \geq (1 - c_2) Z_{n,1}$, and hence, for ensuring that z_h is positive in $[0, t_n]$. Nevertheless, this hypothesis is not needed for verifying that $Z_{0,2} > Z_{0,1} = 0$, and so, it is always ensured the existence of $Z_{1,2}$ (and obviously $Z_{1,1}$), even if the hypothesis does not hold. Hence, if we use another method for checking that $Z_{1,2} \geq (1 - c_2) Z_{1,1}$ (for example, numerically), then it is ensured the existence of $Z_{2,2}$. Repeating this reasoning we can guarantee the existence of $Z_{n,2}$, removing the hypothesis on K .

Following this idea, in the next proposition we state a result analogous to Proposition 3.3 without hypothesis on the kernel.

Proposition 3.4 $\frac{G(y)}{y}$ is unbounded if and only if there exists $H_0 > 0$ such that there are nontrivial collocation solutions in $[0, t_1]$ for $0 < h_0 \leq H_0$. In this case, there always exists $H_1 > 0$ such that there are nontrivial collocation solutions in $[0, t_2]$ for $0 < h_1 \leq H_1$.

Moreover, if $\frac{G(y)}{y}$ is unbounded and there is a **positive** nontrivial collocation solution in $[0, t_n]$, then there exists $H_n > 0$ such that there are nontrivial collocation solutions in $[0, t_{n+1}]$ for $0 < h_n \leq H_n$.

Considering Remark 3.4 and taking the coefficient A given in (12), we can adapt some results of [9] about sufficient conditions on existence for fine meshes and unconditional existence to our new *general conditions*. These results are useful for identifying collocation problems without blow-up.

Proposition 3.5 Let $\frac{G(y)}{y}$ be unbounded near zero, and let the map $t \mapsto K(t, s)$ be continuous in $]s, \epsilon[$ for some $\epsilon > 0$ and for all $0 \leq s < \epsilon$ (this hypothesis can be removed if $c_2 = 1$).

- If K is a convolution kernel and $\frac{G(y)}{y}$ is bounded away from zero, then there is existence for fine meshes.
- If $\frac{y}{G(y)}$ is unbounded away from zero, then there is unconditional existence.

If, in addition, $\frac{G(y)}{y}$ is a strictly decreasing function, then there is at most one nontrivial collocation solution.

Moreover, as in Proposition 3.4, we can state a result about unconditional existence without hypothesis on the kernel:

Proposition 3.6 Let $\frac{G(y)}{y}$ be unbounded near zero.

If $\frac{y}{G(y)}$ is unbounded away from zero and there is a **positive** nontrivial collocation solution in $[0, t_n]$, then there is a nontrivial collocation solution in $[0, t_{n+1}]$ **for any** $h_n > 0$.

If, in addition, $\frac{G(y)}{y}$ is a strictly decreasing function, then there is at most one nontrivial collocation solution.

3.3 Nondivergent existence and uniqueness

Our interest is the study of existence of nontrivial collocation solutions using meshes I_h with arbitrarily small $h > 0$. So, we are not interested in collocation problems whose collocation solutions “escape” to $+\infty$ when a certain $h_n \rightarrow 0^+$, since this is a divergence symptom. Following this criterion, we define the concept of “nondivergent existence”:

Let $0 = t_0 < \dots < t_n$ be a mesh such that there exist nontrivial collocation solutions, and let S_{h_n} be the index set of the nontrivial collocation solutions of the corresponding collocation problem with mesh $t_0 < \dots < t_n < t_n + h_n$. Given $s \in S_{h_n}$, we denote by $Z_{s;n,i}$ the coefficients of the corresponding nontrivial collocation solution verifying equations (3). Then, we say that there is *nondivergent existence* in t_n^+ if

$$\inf_{s \in S_{h_n}} \left\{ \max_{i=1, \dots, m} \{Z_{s;n,i}\} \right\}$$

exists for small enough $h_n > 0$ and it does not diverge to $+\infty$ when $h_n \rightarrow 0^+$. Given a mesh $I_h = \{0 = t_0 < \dots < t_N\}$ such that there exist nontrivial collocation solutions, we say that there is *nondivergent existence* if there is nondivergent existence in t_n^+ for $n = 0, \dots, N$. See [9] for a more detailed analysis, where we also define the concept of *nondivergent uniqueness*.

In [9] it is proved the next result.

Proposition 3.7 *In cases 1 and 2 with existence of nontrivial collocation solutions, there is nondivergent existence if and only if $\frac{G(y)}{y}$ is unbounded near zero.*

If, in addition, G is “well-behaved”, then there is nondivergent uniqueness.

We say that G is “well-behaved” if $\frac{G(\alpha+y)}{y}$ is strictly decreasing near zero for all $\alpha > 0$. Note that this condition is very weak (see [9]).

So, taking into account Propositions 3.1, 3.3 and 3.7, the main result of this paper is:

Theorem 3.1 *(Hypothesis only for case 2 with $c_2 \neq 1$: the map $t \mapsto K(t, s)$ is continuous in $]s, \epsilon[$ for some $\epsilon > 0$ and for all $0 \leq s < \epsilon$.)*

There is nondivergent existence near zero if and only if $\frac{G(y)}{y}$ is unbounded near zero.

If, in addition, G is “well-behaved” then, there is nondivergent uniqueness near zero.

In the same way, we can combine Propositions 3.4 and 3.7 obtaining a result about nondivergent existence and uniqueness without hypothesis on the kernel, and, as Theorem 3.1, it can be useful to study numerically problems with decreasing convolution kernels in case 2 with $c_2 \neq 1$.

We can also combine Propositions 3.2, 3.5 and 3.6 with Proposition 3.7, obtaining results about nondivergent existence and uniqueness (for fine meshes and unconditional) that are useful for identifying collocation problems without blow-up. Some of these results can be reformulated as necessary conditions for the existence of blow-up (see Section 5).

4 Blow-up collocation solutions

In this section we will extend the concept of collocation problem and collocation solution in order to consider the case of “blow-up collocation solutions”.

Definition 4.1 We say that a collocation problem is a *blow-up collocation problem* (or has a blow-up) if the following conditions are held:

- there exists $T > 0$ such that there is no collocation solution in $I = [0, T]$ for any mesh I_h , and
- given $M > 0$ there exists $0 < \tau < T$, and a collocation solution z_h defined on $[0, \tau]$ such that $|z_h(t)| > M$ for some $t \in [0, \tau]$.

We can not speak about “blow-up collocation solutions” in the classic sense, since “collocation solutions” are defined in compact intervals and obviously they are bounded; so, we have to extend first the concept of “collocation solution” to open intervals $I = [0, T[$ before we are in position to define the notion of “blow-up collocation solution”.

Definition 4.2 Let $I := [0, T[$ and I_h be an infinite mesh given by a strictly increasing sequence $\{t_n\}_{n=0}^{+\infty}$ with $t_0 = 0$ and convergent to T .

- A *collocation solution on I* using the mesh I_h is a function defined on I such that it is a collocation solution (in the classic sense) for any finite submesh $\{t_n\}_{n=0}^N$ with $N \in \mathbb{N}$.
- A collocation solution on I is a *blow-up collocation solution* (or has a blow-up) with *blow-up time T* if it is not bounded.

Given a collocation problem with nondivergent uniqueness near zero, a necessary condition for the nondivergent collocation solution to blow-up is that there is neither existence for fine meshes nor unconditional existence. So, for example, given a convolution kernel $K(t, s) = k(t - s)$, in cases 1 and 2 we must require that $\frac{G(y)}{y}$ is unbounded away from zero (moreover,

in case 2 with $c_2 \neq 1$, we must demand that there exists $\epsilon > 0$ such that k is continuous in $]0, \epsilon[$.

For instance, in [5] it is studied equation (1) with convolution kernel $k(t-s) = (t-s)^\beta$, $\beta \geq 0$, and nonlinearity $G(y) = t^\alpha$, $0 < \alpha < 1$, concluding that the nontrivial (exact) solution does not have blow-up. If we consider a collocation problem with the above kernel and nonlinearity in cases 1 and 2, then we can ensure unconditional nondivergent uniqueness (by Propositions 3.2, 3.5 and 3.7); thus, we can also conclude that the nondivergent collocation solution has no blow-up. Actually, we reach the same conclusion considering any kernel satisfying the general conditions (in case 1 and case 2 with $c_2 = 1$) and such that $t \mapsto K(t, s)$ is continuous in $]s, \epsilon[$ for some $\epsilon > 0$ and for all $0 \leq s < \epsilon$ (in case 2 with $c_2 \neq 1$). In Section 5 we discuss in more detail the relationships between necessary conditions for blow-up of collocation solutions and exact solutions.

4.1 Numerical algorithms

First we need to recall the definition of existence near zero, mentioned at the beginning of Section 3, and given in [9].

Definition 4.3 We say that there is **existence near zero** if there exists $H_0 > 0$ such that if $0 < h_0 \leq H_0$ then there are nontrivial collocation solutions in $[0, t_1]$; moreover, there exists $H_n > 0$ such that if $0 < h_n \leq H_n$ then there are nontrivial collocation solutions in $[0, t_{n+1}]$ (for $n = 1, \dots, N-1$ and given $h_0, \dots, h_{n-1} > 0$ such that there are nontrivial collocation solutions in $[0, t_n]$). Note that, in general, H_n depends on h_0, \dots, h_{n-1} .

Given a blow-up collocation problem with nondivergent uniqueness near zero (always in cases 1 and 2), we are going to describe a general algorithm to compute the nontrivial collocation solution and estimate the blow-up time. In each iteration, the stepsize h_n is adapted to H_n , since only if $h_n \leq H_n$ we can ensure the existence of the nontrivial collocation solution.

As long as $h_n \leq H_n$, the solution on t_n for each n is obtained from the attracting fixed point of a certain function $y \mapsto G(\alpha + \beta y)$, given by equation (5) in case 1, or (9) in case 2. Therefore, the key point in the algorithm is to decide whether there is a fixed point or not and, if so, calculate it.

In order to check that there is no fixed point, the most general technique consists on iterate and, if a certain bound (which may depend on the fixed points found in the previous steps) is overcome, then it is assumed that there is no fixed point.

So, if for a certain n there is no fixed point, then a smaller h_n should be taken (e.g. $h_n/2$ in the examples below), and this procedure is repeated with the new α, β corresponding to the new h_n . If h_n becomes smaller than a given tolerance (e.g. 10^{-12} in our examples), the algorithm stops and $t_{n+1} = h_0 + \dots + h_n$ is the estimation of the blow-up time.

Nevertheless, in particular cases, specific algorithms can also be used, as in the case of Example 1, where the fixed points can be found analytically:

- If $0 \leq \alpha < 1$:
 - If $1 - \alpha < \beta \leq \frac{1}{4\alpha}$ the fixed point is $\frac{1-2\alpha\beta-\sqrt{1-4\alpha\beta}}{2\beta^2}$.
 - If $\beta \leq 1 - \alpha$ then the fixed point is $\frac{1}{2} \left(\beta + \sqrt{\beta^2 + 4\alpha} \right)$.
- If $\alpha \geq 1$ and $\beta \leq \frac{1}{4\alpha}$, then the fixed point is $\frac{1-2\alpha\beta-\sqrt{1-4\alpha\beta}}{2\beta^2}$.

If $\beta > \frac{1}{4\alpha}$ then there is no solution for the corresponding h_n . Hence a smaller h_n is taken as described in the general case.

However, both the computational cost and accuracy of the general and specific algorithms are quite similar.

4.2 Examples

We will consider the following examples:

1. $K(t, s) = 1$ and $G(y) = \begin{cases} \sqrt{y} & \text{if } y \in [0, 1] \\ y^2 & \text{if } y \in]1, +\infty[\end{cases}$
2. $K(t, s) = 1$ and $G(y) = \begin{cases} \sqrt{y} & \text{if } y \in [0, 1] \\ e^{y-1} & \text{if } y \in]1, +\infty[\end{cases}$
3. $K(t, s) = k(t - s) = t - s$ and $G(y)$ of Example 1.
4. $K(t, s) = k(t - s) = (\pi(t - s))^{-1/2}$ and $G(y)$ of Example 1.

In the first two examples, the blow-up time of the corresponding (exact) solution is $\hat{t} = 3$, while in the other two the blow-up time is unknown. In [7] it was studied a family of equations to which Example 4 belongs.

In all the examples, both K and G fulfill the *general conditions* and the hypotheses of Theorem 3.1, and thus it is ensured the nondivergent existence and uniqueness near zero (note that since the kernel in Example 4 is decreasing and unbounded near zero, this case can not be studied in [9]). Moreover, $G(y)/y$ is unbounded away from zero and $y/G(y)$ is bounded away from zero, and hence there can exist a blow-up (see Propositions 3.2 and 3.5).

He have found the numerical nondivergent collocation solutions for a given stepsize h ($= h_0$), using the algorithms described in Section 4.1 (the specific for the Example 1 and the general for the rest). In Figures 1, 2, 3 and 4 the estimations of the blow-up times of the collocation solutions for a given stepsize h are depicted, varying c_1 in case 1 or c_2 in case 2. Note that the different graphs for different h intersect each other in a fairly good approximation of the blow-up time, as it is shown in Tables 1, 2, 4 and 5.

In Examples 3 and 4 we do not know the exact value of the blow-up time. However, in order to make a study of the relative error analogous to the previous examples, we have taken as blow-up time for Example 3 the approximation for $h = 0.001$ in case 2 with $c_2 = 0.5$: $t = 5.78482$; and for Example 4 the approximation for $h = 0.0001$ in case 2 with $c_2 = 2/3$ (*Radau I* collocation points): $t = 1.645842$. Results are shown in Tables 7, 8, 10 and 11.

The relative error varying c_1 (case 1) or c_2 (case 2) is the “relative vertical size” of the graph, and it decreases at the same rate as h . On the other hand, the relative error of the intersection decreases faster, sometimes at the same rate as h^2 . Moreover, in case 1, the best approximations are reached with $c_1 \approx 0.5$, and in case 2 with $c_2 \approx 2/3$ (approximately *Radau I*) for Examples 1, 2 and 4 (see Tables 3, 6 and 12), while for Example 3 the best approximations are reached with $c_2 \approx 0.5$; however, in Table 9 are also shown the approximations and their corresponding errors for the *Radau I* collocation points. On the other hand, the intersections method offers better results, but at a greater computational cost.

5 Discussion and comments

The main necessary conditions for a collocation problem to have a blow-up are obtained from Propositions 3.2 (case 1) and 3.5 (case 2), and are mostly related to the nonlinearity, since the assumption on the kernel “the map $t \mapsto K(t, s)$ is continuous in $]s, \epsilon[$ for some $\epsilon > 0$ and for all $0 \leq s < \epsilon$ ” is only required in case 2 with $c_2 \neq 1$ and it is a very weak hypothesis. Hence, assuming that the kernel satisfies this hypothesis, the main necessary condition for the existence of a blow-up is:

1. $\frac{y}{G(y)}$ is bounded away from zero.

In addition, for convolution kernels, there is another necessary condition:

2. $\frac{G(y)}{y}$ is unbounded away from zero, i.e. there exists a sequence $\{y_n\}_{n=1}^{+\infty}$ of positive real numbers and divergent to $+\infty$ such that $\lim_{n \rightarrow +\infty} \frac{y_n}{G(y_n)} = 0$.

In [4] it is given a necessary and sufficient condition for the existence of blow-up (exact) solutions for equation (1) with a kernel of the form $K(t, s) = (t - s)^{\alpha-1}r(s)$ with $\alpha > 0$, r nondecreasing and continuous for $s \neq 0$, $r(s) = 0$ for $s \leq 0$, and $r(s) > 0$ for $s > 0$:

$$\int_{\delta}^{+\infty} \left(\frac{s}{G(s)} \right)^{1/\alpha} \frac{ds}{s} < +\infty, \quad \delta > 0. \quad (13)$$

This also holds for convolution kernels of Abel type $K(t, s) = (t - s)^{\alpha-1}$ with $\alpha > 0$, generalizing some results given in [3]. Next we will show that necessary conditions 1 and 2 above mentioned are also necessary conditions for the integral given in (13) to be convergent, and thus for the existence of a blow-up (exact) solution.

Proposition 5.1 *If (13) holds, then $\frac{G(y)}{y}$ is unbounded away from zero.*

Proof. Let us suppose that $\frac{G(y)}{y}$ is bounded away from zero. So, there exists $M > 0$ such that $\frac{G(y)}{y} < M$ for all $y > \delta$. Hence

$$\int_{\delta}^{+\infty} \left(\frac{s}{G(s)} \right)^{1/\alpha} \frac{ds}{s} > \left(\frac{1}{M} \right)^{1/\alpha} \int_{\delta}^{+\infty} \frac{ds}{s} = +\infty.$$

■

Proposition 5.2 *If (13) holds, then $\frac{y}{G(y)}$ is bounded away from zero.*

Proof. By Proposition 5.1, $\frac{G(y)}{y}$ is unbounded away from zero and hence, there exists a strictly increasing sequence $\{y_n\}_{n=1}^{+\infty}$ with $y_1 > \delta$ and divergent to $+\infty$ such that $\frac{y_n}{G(y_n)} < \frac{1}{2^\alpha}$ for all n .

Let us suppose that $\frac{y}{G(y)}$ is unbounded away from zero; so, we can choose $\{y_n\}_{n=1}^{+\infty}$ such that there exist $y'_n \in]y_n, y_{n+1}[$ with $\frac{y'_n}{G(y'_n)} = 1$ for each n . Moreover, since the nonlinearity G is positive and strictly increasing, we have that $\frac{y'_n}{G(y_n)} > \frac{y'_n}{G(y'_n)} = 1$, and then $y'_n > G(y_n)$. Hence, we have

$$\begin{aligned} \int_{y_n}^{y_{n+1}} \left(\frac{s}{G(s)} \right)^{1/\alpha} \frac{ds}{s} &> \int_{y_n}^{y'_n} \left(\frac{s}{G(s)} \right)^{1/\alpha} \frac{ds}{s} > \int_{y_n}^{y'_n} \left(\frac{s}{G(y'_n)} \right)^{1/\alpha} \frac{ds}{s} \\ &= \alpha \left(1 - \left(\frac{y_n}{y'_n} \right)^{1/\alpha} \right) > \alpha \left(1 - \left(\frac{y_n}{G(y_n)} \right)^{1/\alpha} \right) > \frac{\alpha}{2}. \end{aligned}$$

Therefore, (13) does not hold.

■

An interesting problem which is not fully resolved is to continue clarifying the relationship between the existence of blow-up in exact solutions and in collocation solutions.

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A Figures and tables

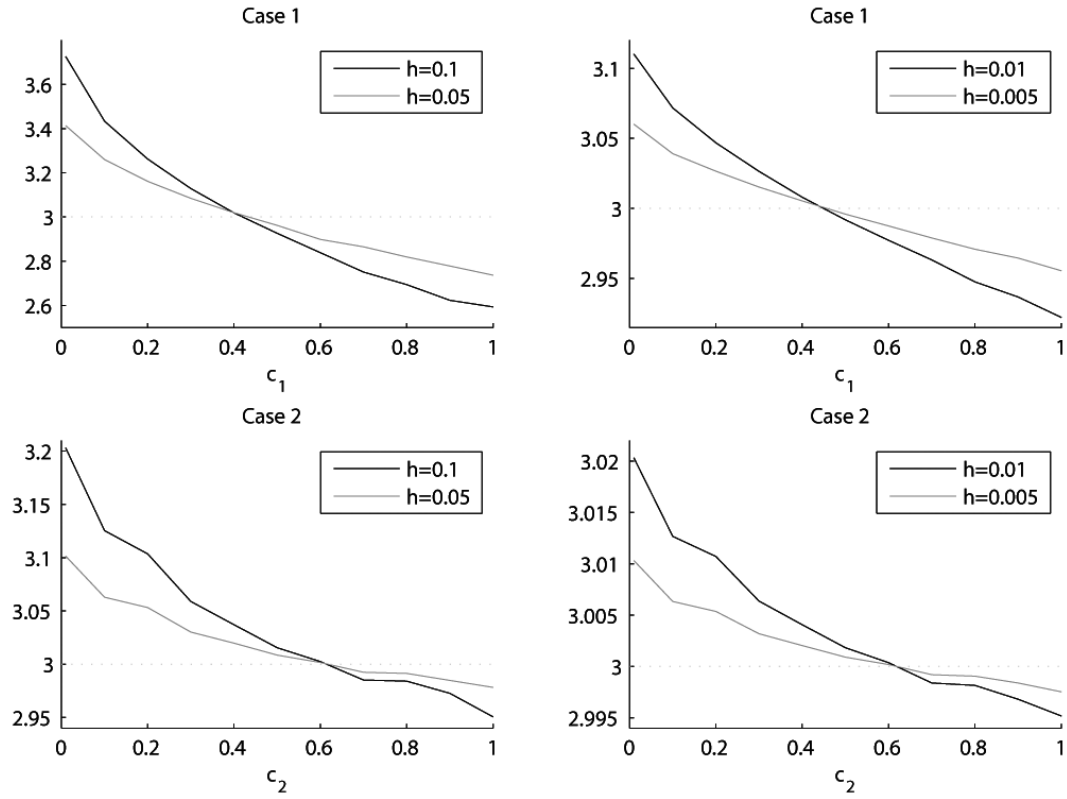


Figure 1: Example 1. Numerical estimation of the blow-up time of collocation solutions varying c_1 (case 1) or c_2 (case 2), for different stepsizes h . The blow-up time of the corresponding (exact) solution is $\hat{t} = 3$.

Case 1 ($m = 1, c_1 > 0$)	Varying c_1			Intersection	
	min. t	max. t	Rel. error	t	Rel. error
$h = 0.1$	2.60 ($c_1 = 1$)	3.66 ($c_1 = 0.01$)	$4 \cdot 10^{-1}$	3.03 ($c_1 = 0.39$)	10^{-2}
$h = 0.05$	2.78 ($c_1 = 1$)	3.40 ($c_1 = 0.01$)	$2 \cdot 10^{-1}$		
$h = 0.01$	2.92 ($c_1 = 1$)	3.11 ($c_1 = 0.01$)	$6 \cdot 10^{-2}$	3.002 ($c_1 = 0.44$)	$7 \cdot 10^{-4}$
$h = 0.005$	2.96 ($c_1 = 1$)	3.06 ($c_1 = 0.01$)	$3 \cdot 10^{-2}$		

Table 1: Example 1. Numerical data of Figure 1 (case 1).

Case 2 ($m = 2, c_1 = 0$)	Varying c_2			Intersection	
	min. t	max. t	Rel. error	t	Rel. error
$h = 0.1$	2.95 ($c_2 = 1$)	3.20 ($c_2 = 0.01$)	$8 \cdot 10^{-2}$	3.0003 ($c_2 = 0.6178$)	10^{-4}
$h = 0.05$	2.98 ($c_2 = 1$)	3.10 ($c_2 = 0.01$)	$4 \cdot 10^{-2}$		
$h = 0.01$	2.995 ($c_2 = 1$)	3.020 ($c_2 = 0.01$)	$8 \cdot 10^{-3}$	3.000003 ($c_2 = 0.6232$)	10^{-6}
$h = 0.005$	2.998 ($c_2 = 1$)	3.010 ($c_2 = 0.01$)	$4 \cdot 10^{-3}$		

Table 2: Example 1. Numerical data of Figure 1 (case 2).

	Case 1 - $c_1 = 0.5$		Case 2 - Radau I	
	Blow-up	Rel. error	Blow-up	Rel. error
$h = 0.1$	2.933883	$2.2 \cdot 10^{-2}$	2.995253	$1.6 \cdot 10^{-3}$
$h = 0.05$	2.965002	$1.2 \cdot 10^{-2}$	2.997602	$8 \cdot 10^{-4}$
$h = 0.01$	2.992885	$2.4q \cdot 10^{-3}$	2.999519	$1.6 \cdot 10^{-4}$
$h = 0.005$	2.996434	$1.2 \cdot 10^{-3}$	2.999759	$8 \cdot 10^{-5}$

Table 3: Example 1. Numerical estimations and relative errors of the blow-up time of collocation solutions in case 1 with $c_1 = 0.5$, and case 2 with *Radau I* collocation points, $c_1 = 0$, $c_2 = 2/3$, for different stepsizes h .

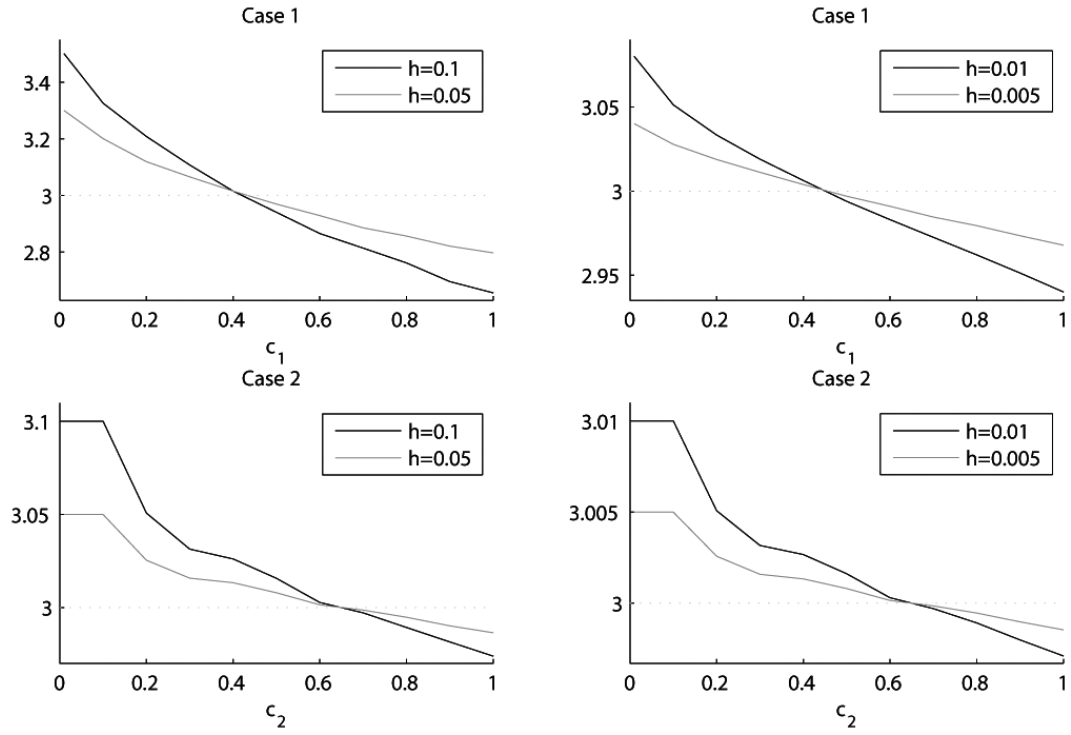


Figure 2: Example 2. Numerical estimation of the blow-up time of collocation solutions varying c_1 (case 1) or c_2 (case 2), for different stepsizes h . The blow-up time of the corresponding (exact) solution is $\hat{t} = 3$.

Case 1 ($m = 1, c_1 > 0$)	Varying c_1			Intersection	
	min. t	max. t	Rel. error	t	Rel. error
$h = 0.1$	2.68 ($c_1 = 1$)	3.50 ($c_1 = 0.01$)	$3 \cdot 10^{-1}$	3.015 ($c_1 = 0.399$)	$5 \cdot 10^{-3}$
$h = 0.05$	2.82 ($c_1 = 1$)	3.30 ($c_1 = 0.01$)	$1.5 \cdot 10^{-1}$		
$h = 0.01$	2.94 ($c_1 = 1$)	3.08 ($c_1 = 0.01$)	$4.6 \cdot 10^{-2}$	3.0008 ($c_1 = 0.444$)	$2.6 \cdot 10^{-4}$
$h = 0.005$	2.98 ($c_1 = 1$)	3.04 ($c_1 = 0.01$)	$2.3 \cdot 10^{-2}$		

Table 4: Example 2. Numerical data of Figure 2 (case 1).

Case 2 ($m = 2, c_1 = 0$)	Varying c_2			Intersection	
	min. t	max. t	Rel. error	t	Rel. error
$h = 0.1$	2.97 ($c_2 = 1$)	3.10 ($c_2 = 0.01$)	$4.3 \cdot 10^{-2}$	3.00012 ($c_2 = 0.6466$)	$4 \cdot 10^{-5}$
$h = 0.05$	2.99 ($c_2 = 1$)	3.05 ($c_2 = 0.01$)	$2 \cdot 10^{-2}$		
$h = 0.01$	2.997 ($c_2 = 1$)	3.010 ($c_2 = 0.01$)	$4.3 \cdot 10^{-3}$	3.000011 ($c_2 = 0.65114$)	$3.6 \cdot 10^{-6}$
$h = 0.005$	2.9985 ($c_2 = 1$)	3.005 ($c_2 = 0.01$)	$2.15 \cdot 10^{-3}$		

Table 5: Example 2. Numerical data of Figure 2 (case 2).

	Case 1 - $c_1 = 0.5$		Case 2 - Radau I	
	Blow-up	Rel. error	Blow-up	Rel. error
$h = 0.1$	2.941795	$2 \cdot 10^{-2}$	2.999559	$1.5 \cdot 10^{-4}$
$h = 0.05$	2.970341	10^{-2}	2.999778	$7.4 \cdot 10^{-5}$
$h = 0.01$	2.993979	$2 \cdot 10^{-3}$	2.999955	$1.5 \cdot 10^{-5}$
$h = 0.005$	2.996983	10^{-3}	2.999978	$7.3 \cdot 10^{-6}$

Table 6: Example 2. Numerical estimations and relative errors of the blow-up time of collocation solutions in case 1 with $c_1 = 0.5$, and case 2 with *Radau I* collocation points, $c_1 = 0$, $c_2 = 2/3$, for different stepsizes h .

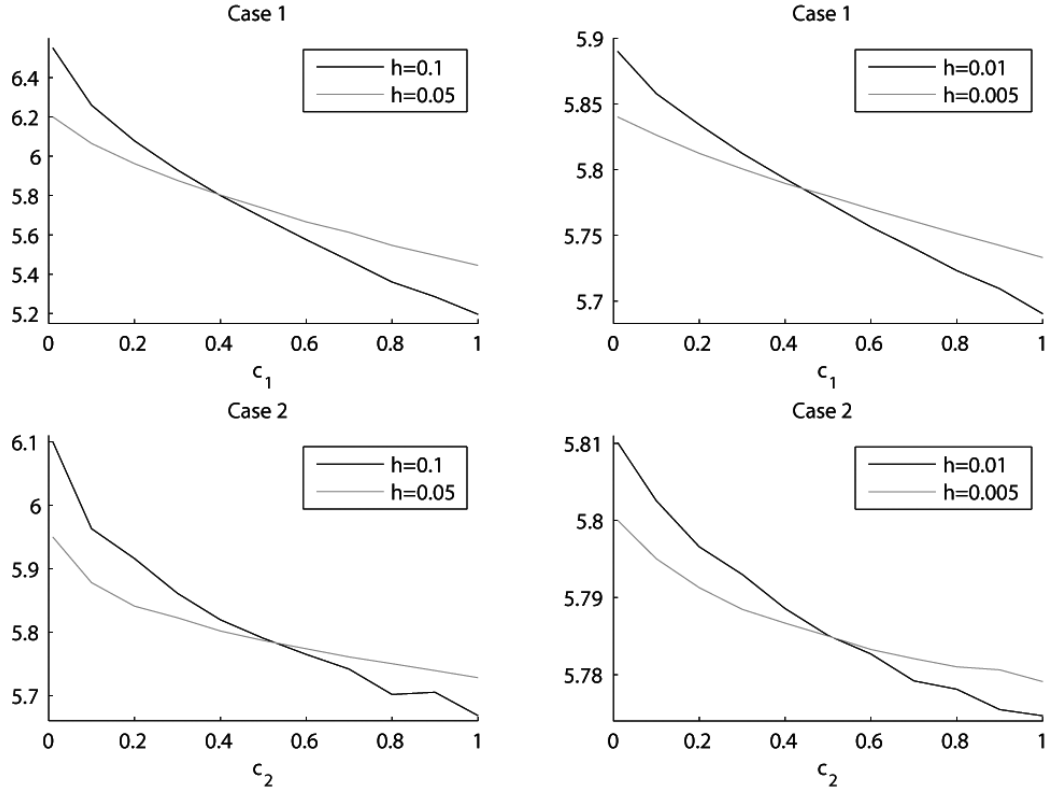


Figure 3: Example 3. Numerical estimation of the blow-up time of collocation solutions varying c_1 (case 1) or c_2 (case 2), for different stepsizes h .

Case 1 ($m = 1, c_1 > 0$)	Varying c_1			Intersection	
	min. t	max. t	Rel. error	t	Rel. error
$h = 0.1$	5.20 ($c_1 = 1$)	6.55 ($c_1 = 0.01$)	$2.3 \cdot 10^{-1}$	5.8055 ($c_1 = 0.3961$)	$3.6 \cdot 10^{-3}$
$h = 0.05$	5.45 ($c_1 = 1$)	6.20 ($c_1 = 0.01$)	$1.3 \cdot 10^{-1}$		
$h = 0.01$	5.71 ($c_1 = 1$)	5.89 ($c_1 = 0.01$)	$3.1 \cdot 10^{-2}$	5.7856 ($c_1 = 0.441$)	$1.4 \cdot 10^{-4}$
$h = 0.005$	5.735 ($c_1 = 1$)	5.840 ($c_1 = 0.01$)	$1.8 \cdot 10^{-2}$		

Table 7: Example 3. Numerical data of Figure 3 (case 1). The blow-up time of the corresponding (exact) solution is assumed to be $\hat{t} \approx 5.78482$.

Case 2 ($m = 2, c_1 = 0$)	Varying c_2			Intersection	
	min. t	max. t	Rel. error	t	Rel. error
$h = 0.1$	5.67 ($c_2 = 1$)	6.10 ($c_2 = 0.01$)	$7.4 \cdot 10^{-2}$	5.78304 ($c_2 = 0.5286$)	$3 \cdot 10^{-4}$
$h = 0.05$	5.73 ($c_2 = 1$)	5.95 ($c_2 = 0.01$)	$3.8 \cdot 10^{-2}$		
$h = 0.01$	5.775 ($c_2 = 1$)	5.810 ($c_2 = 0.01$)	$6 \cdot 10^{-3}$	5.7848 ($c_2 = 0.514$)	$3.5 \cdot 10^{-6}$
$h = 0.005$	5.779 ($c_2 = 1$)	5.800 ($c_2 = 0.01$)	$3.6 \cdot 10^{-3}$		

Table 8: Example 3. Numerical data of Figure 3 (case 2). The blow-up time of the corresponding (exact) solution is assumed to be $\hat{t} \approx 5.78482$.

	Case 1 - $c_1 = 0.5$		Case 2 - Radau I		Case 2 - $c_2 = 0.5$	
	Blow-up	Rel. error	Blow-up	Rel. error	Blow-up	Rel. error
$h = 0.1$	5.694865	$1.6 \cdot 10^{-2}$	5.751797	$5.7 \cdot 10^{-3}$	5.788215	$5.9 \cdot 10^{-4}$
$h = 0.05$	5.739117	$7.9 \cdot 10^{-3}$	5.767963	$2.9 \cdot 10^{-3}$	5.786612	$3.1 \cdot 10^{-4}$
$h = 0.01$	5.775121	$1.7 \cdot 10^{-3}$	5.781037	$6.5 \cdot 10^{-4}$	5.784995	$3 \cdot 10^{-5}$
$h = 0.005$	5.780132	$8.1 \cdot 10^{-4}$	5.783066	$3 \cdot 10^{-4}$	5.784875	$9.5 \cdot 10^{-6}$

Table 9: Example 3. Numerical estimations and relative errors of the blow-up time of collocation solutions in case 1 with $c_1 = 0.5$, and case 2 with *Radau I* collocation points, $c_1 = 0, c_2 = 2/3$, and $c_1 = 0, c_2 = 0.5$, for different stepsizes h . The blow-up time of the corresponding (exact) solution is assumed to be $\hat{t} \approx 5.78482$.

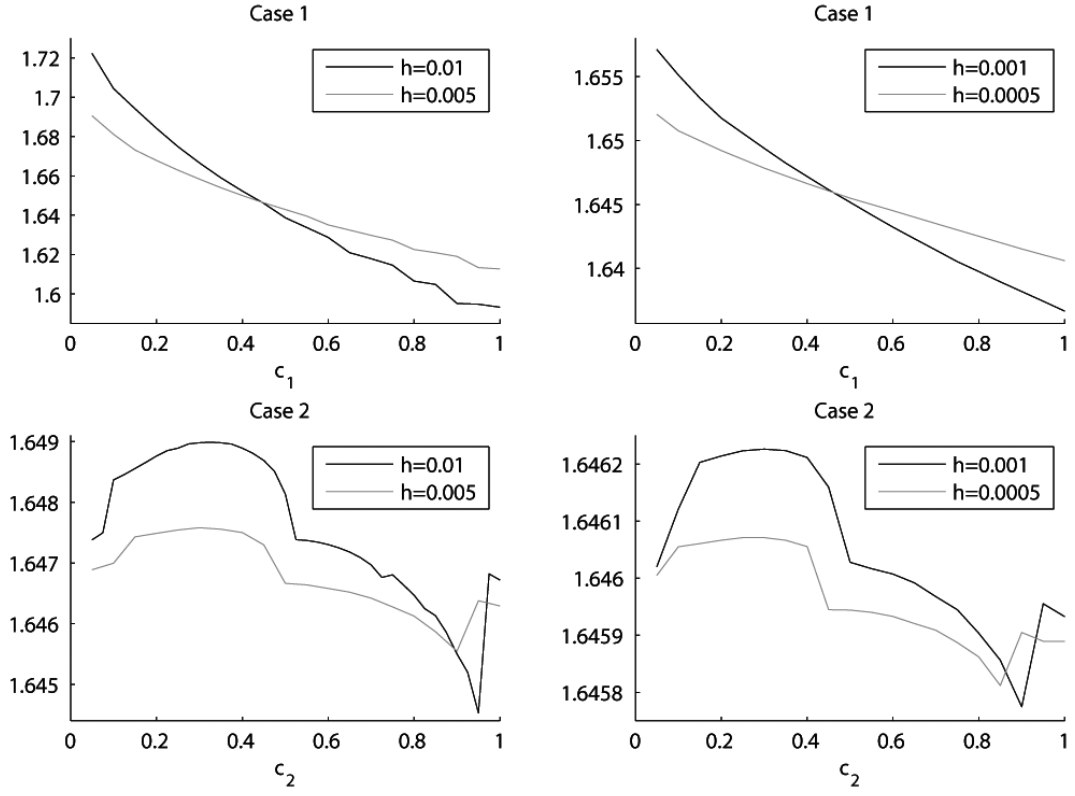


Figure 4: Example 4. Numerical estimation of the blow-up time of collocation solutions varying c_1 (case 1) or c_2 (case 2), for different stepsizes h .

Case 1 ($m = 1, c_1 > 0$)	Varying c_1			Intersection	
	min. t	max. t	Rel. error	t	Rel. error
$h = 0.01$	1.593 ($c_1 = 1$)	1.722 ($c_1 = 0.05$)	$7.8 \cdot 10^{-2}$	1.6467 ($c_1 = 0.444$)	$5.2 \cdot 10^{-4}$
$h = 0.005$	1.613 ($c_1 = 1$)	1.691 ($c_1 = 0.05$)	$4.7 \cdot 10^{-2}$		
$h = 0.001$	1.6367 ($c_1 = 1$)	1.6571 ($c_1 = 0.05$)	$1.2 \cdot 10^{-2}$	1.64595 ($c_1 = 0.461$)	$6.6 \cdot 10^{-5}$
$h = 0.0005$	1.6406 ($c_1 = 1$)	1.6521 ($c_1 = 0.05$)	$7 \cdot 10^{-3}$		

Table 10: Example 4. Numerical data of Figure 4 (case 1). The blow-up time of the corresponding (exact) solution is assumed to be $\hat{t} \approx 1.645842$.

Case 2 ($m = 2, c_1 = 0$)	Varying c_2		
	min. t	max. t	Rel. error
$h = 0.01$	1.6445 ($c_2 = 0.95$)	1.6490 ($c_2 = 0.33$)	$2.7 \cdot 10^{-3}$
$h = 0.005$	1.6456 ($c_2 = 0.9$)	1.6476 ($c_2 = 0.3$)	$1.2 \cdot 10^{-3}$
$h = 0.001$	1.64577 ($c_2 = 0.9$)	1.64622 ($c_2 = 0.3$)	$2.7 \cdot 10^{-4}$
$h = 0.0005$	1.64581 ($c_2 = 0.85$)	1.64607 ($c_2 = 0.28$)	$1.6 \cdot 10^{-4}$

Table 11: Example 4. Numerical data of Figure 4 (case 1). The blow-up time of the corresponding (exact) solution is assumed to be $\hat{t} \approx 1.645842$.

	Case 1 - $c_1 = 0.5$		Case 2 - Radau I	
	Blow-up	Rel. error	Blow-up	Rel. error
$h = 0.01$	1.638700	$4.3 \cdot 10^{-3}$	1.646991	$7 \cdot 10^{-4}$
$h = 0.005$	1.642843	$1.8 \cdot 10^{-3}$	1.646540	$4.2 \cdot 10^{-4}$
$h = 0.001$	1.645172	$4.1 \cdot 10^{-4}$	1.645985	$8.7 \cdot 10^{-5}$
$h = 0.0005$	1.645491	$2.1 \cdot 10^{-4}$	1.645919	$4.7 \cdot 10^{-5}$

Table 12: Example 4. Numerical estimations and relative errors of the blow-up time of collocation solutions in case 1 with $c_1 = 0.5$, and case 2 with *Radau I* collocation points, $c_1 = 0, c_2 = 2/3$, for different stepsizes h . The blow-up time of the corresponding (exact) solution is assumed to be $\hat{t} \approx 1.645842$.